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On the physical aspects of Henry Stommel's theory of intensification of western boundary currents and Ekman spiral of currents

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Abstract

Henry Stommel arrived at vorticity equation of subtropical gyre of the wind-driven oceanic currents, by taking into consideration of the frictional force and coriolis force and also by following the general convention of direction of frictional force. He was the first scientist to discover the importance of planetary vorticity to intensify the western boundary currents of the gyre. His 2- dimensional, horizontal, theory in this regard is described in simple physical terms in this paper.

Considering the same two forces, Walfrid Ekman framed in his own way 1-dimentional vertical variation of wind-driven horizontal currents veering to the right in N.H. and decreasing exponentially with depth. The system of currents is known as Ekman spiral. The depth above which only significant currents occur is called by Ekman the depth of frictional resistance. The surface current of the spiral makes 45° to the right of wind flow.

Stommel's model of intensification of western boundary currents and Ekman's model of vertically spiral horizontal currents are dealt with here to bring out their merits and limitations, if any, from the physical point of view with the aim of stimulating the young oceanographers. As the vertically integrated horizontal transport of water from the Ekman spiral makes 90° to the wind, near shore upwelling, where the wind blows along the coast with the land mass to the left of wind, is conveniently attributed to the transport from the spiral. As the transport has its origin in one dimensional vertical spiral of currents, it would not have a measurable or estimable direct effect on upwelling which is practically 3-dementional covering over wider area.

Introduction

Winds and ocean currents, in general, are governed by geostrophic balance of forces on a level surface in air for winds and in ocean for currents. Horizontal motion under balance of different sets of forces is dealt with in detail by Hess (1959), However, they may be briefly stated here. The forces involved in the balance are pressure gradient force $-1/\rho\ \partial p/\partial r$ and the coriolis force fV for a steady flow; p stands for pressure, ρ for density, r for lineal distance along which the pressure gradient is measured, V for velocity and f for coriolis parameter, $2\ \omega\ \sin\phi$, where ω is earth's angular velocity and ϕ is latitude of the place at which the motion is considered. Pressure gradient force acts towards low pressure and coriolis force at right angles towards right of direction of flow in the northern hemisphere. If the flow is cyclonic, centrifugal force (V^2/r) which acts away from centre *i.e.* in the direction of increasing r comes into play in addtion to pressure gradient force and coriolis force. Cyclonic or anticyclonic flow is known as gradient flow. There is another system of flow known as cyclostrophic flow which is relatively of small scale and in low latitudes where f is small. Examples of cyclosprophic flow are dust levels, tornados and water spouts in ocean. This is governed by pressure gradient force and centrifugal force in balance to produce steady flow ($-1/p\ {\partial p}/{\partial r} + V^2/r$). In this case the flow is always around a low pressure and the rotation can be either clockwise or anticlockwise. This conclusion is drawn by the characteristic nature of centrifugal force which acts on particle or parcel in motion in the direction of increasing r (away from centre) and therefore pressure gradient force should act towards the centre (towards decreasing r) to balance the other force, which again is possible in the case of low pressure system at the centre. If the flow is balanced by coriolis force fV and centrifugal force V^2/r, the flow is known as inertial flow. In this case the balance is possible only the clockwise rotating flow. In the reverse flow, balancing is not possible. Significance of inertial flow lies in its time period. $T = 2\pi r/v = 2\pi/f = \pi/\omega\sin\phi$ which is called the half pendulum day (Sverdrup *et.al.*, 1942).

Equations of balanced forces working on one gram of mass to produce steady flow may be summarised as follows:

$-1/\rho\ {\partial p}/{\partial r} + fv = 0$ (geostrophic flow)

$-1/p\ {\partial p}/{\partial r} + fv + V^2/r = 0$ (Gradient flow)

$-1/p\ {\partial p}/{\partial r} + V^2/r = 0$ (Cyclostrophic flow)

$fv + V^2/r = 0$ (inertial flow)

Two systems of wind-driven motion of sea water with rotational tendency of fluid (sea water) are given special treatments, separately by Henry Stommel (1948) and Walfrid Ekman (1902) taking the coriolis force and frictional force into account to produce horizontal steady velocities of motion and the authors have given different treatment in their own style. In both the cases, the surface currents are generated by wind stress.

Stommel's model of intensification of western boundary currents and Ekman's model of vertically spiraling horizontal currents are dealt with here to bring out their merits and limitations, if any, from the physical point of view with the aim of stimulating the young oceanographers to initiate to take the subject further.

I remember with gratitude Dr. V.V.R. Varadachari who retired as Director from the National Institute of Oceanography (CSIR), Goa. I am thankful to Prof. (Retd.) Dr. D.R.K.S. Rao for helpful suggestions in vector analysis. Thanks are also due to Mrs. K. Sugatri Devi, Asst. Librarian, Andhra University for her library service with personal interest.

Intensification of Western Boundary currents (Stommel's Theory)

The winds, the equatorial easterlies (below 20° to equator) and the subtropical westerlies in 40's offer the wind stress at the ocean surface to generate surface currents in the form of a gyre called subtropical gyre in each hemisphere in each of the major oceans (Atlantic, Pacific and South Indian Ocean). Therefore, the subtropical gyre is a wind driven ocean circulation. Henry Stommel (1948) considered the vertically integrated vorticity equation to eliminate pressure from equations of motion and is given by βv + curl (τ=F) when β is latitudinal change of coriolis coefficient f, V is the poleward volume transport, r is the wind stress and F is the frictional force. The left hand side of the above equation is called planetary vorticity. The wind stress curl is almost uniform throughout the ocean. Adopting the general convention of direction of frictional force in the opposite direction of velocity, Stommel explained for the first time "how and why" the western boundary currents are intensified.

The physical concept of Stommel's explanation may be put in its pictorial form as follows:

Take the western wing of the subtropical gyre. The currents are northward increasing as they appraoch the coast. The frictional force is directed southward (in the opposite direction of current). The system of frictional force is such that anticlockwise vorticity is developed (See Fig.1). It may be noticed that the curl $\delta fv/\delta y$ or $\delta F/\delta x$ is represented by curved arrow raising from low to high value of force, in other words, from short to long arrow representing force. As coriolis force increases with increasing latitude (sin f), and to balance with this increasing planetary vorticity, F should increase and for this purpose V should increase. As these two vorticities are opposite to one another individual components can increase enormously as this would not lead to correspondingly enormous growth of net vorticity. Thus V can increase. Coming to the right wing of the gyre, the velocity of the flow is southward (toward equator, followed by less coriolois force as sinϕ decreases toward equator. The planetary vorticity therefore is anticlockwise (see Fig.1). Vorticity due to frictional force is also anticlockwise. As the vorticities are in the same phase (anticlockwise) there is no chance for the components of vorticity to attain enormous values. Therefore, the southward (equator ward) flowing eastern boundary current meets with a limit in growth, hence, relatively weak when compared to its counterpart which is in the western boundary. In other words, the western boundary currents are

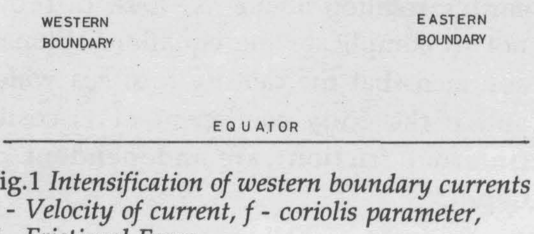

Fig.1 *Intensification of western boundary currents*
v - Velocity of current, f - coriolis parameter,
F - Frictional Force
(Vorticities of f, and F are schematically indicated in figure).

intesified. Stommel (1948) gained the credit of being the first scientist (Fairbridge, 1966) to recognise the worth of coriolis force to intensify the western boundary currents.

Ekman's model of spiraling currents with depth

Prof. Walfrid Ekman (1902) derived inspiration to develop a mathematical model of currents to explain Fridtjof Nansen's observation in the Polar Sea that ice drifted to 20° to 40° to the right of the wind (Sverdrup *et.al.*,1942). In his model Ekman took into consideration the coriolis force and frictional force caused by vertical gradients of horizontal steady velocities to balance in equation. Let x, y and z be rectangular reference axes fixed at the surface of the sea at latitude Φ greater than zero (equator) in the Northern Hemisphere, z directing vertically down, x towards east and y towards north. Let u and v the horizonal components of current resolved into z and y directions respectively. The equations of forces balanced may be written as :

$$0 = fv\ 1/\rho\ u\ \partial^2 u/\partial z^2 \quad (1)$$

$$0 = -fu+1/p\ \mu\ \partial^2 v/\partial z^2 \quad (2)$$

Where f is coriolis parameter (f=2 $\omega\sin\Phi$,ω being the angular velocity of earth's rotation about its axis. In order not to complicate the equations, Ekman assumed that the density ρ of sea water and μ the eddy coefficient of viscosity (internal friction) are independent of depth,z.

Adding eqs. (1) and (2) vectorially,

$$0 = -if\ (u+iv)+\ 1/\rho\mu\ \delta^2\ (u+iv)\delta z^2 \quad ....(3)$$

Where i = $\sqrt{-1}$

$$\text{i.e.}O=\partial^2(u+iv)\partial z^2 + i^2 i\ \rho\ f/\mu\ (u+iv) \quad ....(4)$$

Representing $\rho f/2\mu$ by n for simplicity

$$0 = \partial^2(u+iv)/\partial z^2 + i^2 i 2n\ (u+iv) \quad ......(5)$$

The above equation is a second order and linear (first degree) equation in (u=iv) and z as independent variable. It is an equation of Simple Harmonic Motion (SHM) like $\partial^2 x/\partial t^2+\omega^2 x = 0$, where x is linear displacement of a particle in SHM, t is time and w is a consentant (not the angular velocity of earth's rotation about its axis). The solution for the above is

$$x = A\ \exp\ (\omega t) + B\ \exp\ (-\omega t) \quad ....(6)$$

From equation (5)

$$\omega^2 = i^2\ i2n$$

$$\omega = i\ \sqrt{i}\ \sqrt{2n} = (i-1)\ \sqrt{n} \quad ....(7)$$

since $(1+i)^2 = 2i$, $\sqrt{i} = 1/\sqrt{2}+i/\sqrt{2}$

It may be noticed that the very imaginary number contained two parts namely imaginary and also real. Therefore, the solution for eq. (5) is

$$(u+iv) = A\ \exp\ \sqrt{n}\ (i-1)z + B\ \exp - \sqrt{n}\ (i-1)\ Z \quad ....(8)$$

at z = ∞ , u = iv = 0, and this leads to B = 0

Therefore, $(u + iv) = A\ \exp \sqrt{n}\ (i-1)\ z$ ....(9)

The wind vector and its stress are assumed towards north

Let (u+iv) at z =0 be u_o. Eq. (9) becomes

$$(u+iv = u_o\ \exp\ \sqrt{n}\ (i-1)z$$

$$\text{i.e. } u+iv = u_o\ \exp - \sqrt{n}z\ X\ \exp\ i\ \sqrt{n}z$$

$= u_o \exp - \sqrt{n}z (\cos \sqrt{n}z + i \sin \sqrt{n}z)$ ....(10)

Now separating the components of velocity

$$u = u_o \exp - \sqrt{n}z \cos \sqrt{n}z$$
$$v = u_o \exp - \sqrt{n}z \sin \sqrt{n}z \quad ....(11)$$

Where, as stated earlier $n = \rho f / 2u$

To convert $f$nz into angle:

Angle measured in anticlockwise direction with respect to x axis is considered positive and clockwise negative. The angle θ can be derived from $\tan \theta = v/u = \partial v / \partial u$ and angle θ at the surface is $\pi/4$ or 45°. With reference to this, below the surface the angle of the veering current is negative. At great depths beyond z = D, the velocities are negligible. The fractional value of π at z is equal to $\pi z/D$, if π be the angle at depth D. With this notation $\sqrt{n} = \pi/D$. Sverdrup has written (Sverdrup *et. al.*1942)

$\pi (\mu / p\ \omega \sin \Phi) = D$ and this also leads to $\sqrt{n} = \pi/D$ and $\sqrt{n}z = \pi z/D$.

Therefore the angle at z is given by (n/4 - nz/D).

Now eq. 9 becomes

$$\mu_z = u_o \exp(-\pi z/D) \cos(\pi/4 - \pi z/D) \quad ....(12)$$

$$v_z = u_o \exp(-\pi z/D) \sin(\pi/4 - \pi z/D) \quad ....(13)$$

$$\text{and } D = \pi(\mu/\rho\omega \sin \Phi) \quad .....(14)$$

Ekman model is schematically represented in Fig. (2). The figure is self explanatory. The hodograph, the end points of the current vectors projected on a horizontal plane, is also presented in the

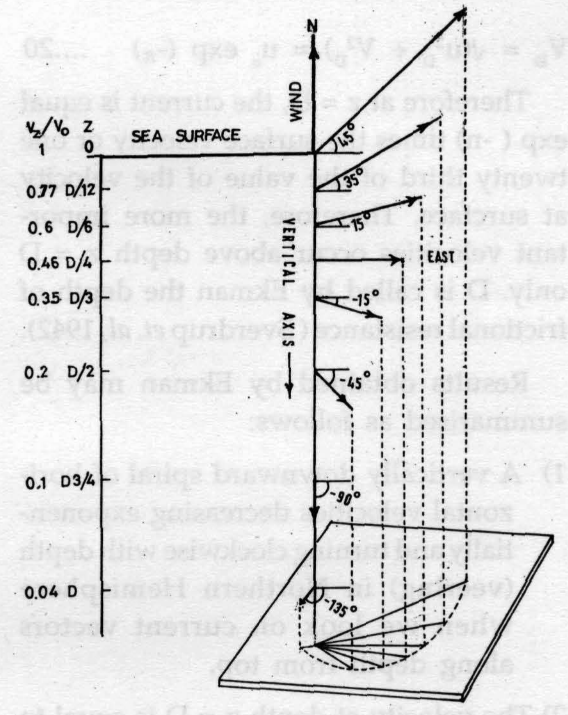

Fig.2 Ekman spiral (veering currents) in N. H. with hodograph

Horizontal current velocities, at different depths, as a ratio to that at surface Vz/Vo. Angle of current direction is (n/4 - nz/D) at depth z. Wind blows towards north.

figure. The hodograph forms into a logarithmic spiral. By the time the current turns to east at depth z = D/4, its magnitude reduces to half of its surface value. Significance of depth, D, may be appreciated by the following treatment.

As z attains D value,

$$u_D = u_o \exp(-\pi) \cos(\pi/4 - \pi) \quad ....15$$

$$V_D = u_o \exp(-\pi) \sin(\pi/4 - \pi) \quad ....16$$

$$\text{ie} \quad u_D = -u_o (1 - \sqrt{2}) \exp(-\pi) \quad ....17$$

$$v_D = -u_o (1 - \sqrt{2}) \exp(-\pi_2) \quad ....18$$

$$u^2_D + V^2_D = u^2_o (\exp - \pi)^2 \quad ....19$$

If VD = is current vector at depth D,

$V_D = \sqrt{(u^2_D + V^2_D)} = u_o \exp(-\pi)$ 20

Therefore at z = D, the current is equal exp (-n) times the surface velocity or one twenty third of the value of the velocity at surcface. Therefore, the more important velocities occur above depth z = D only. D is called by Ekman the depth of frictional resistance (Sverdrup *et. al*, 1942).

Results obtained by Ekman may be summarised as follows:

1) A vertically downward spiral of horizontal velocities decreasing exponentially and turning clockwise with depth (veering) in Northern Hemisphere when we look on current vectors along depth from top.

2) The velocity at depth z = D is equal to exp (-n) times the surface velocity or one twenty third of velocity at the surface. Therefore, more significant velocities occur above the depth z = D which is called the "depth of frictional resistance".

3) The surface current of the spiral remained at 45° to the right of the flow of wind. This result offers the reason for ice drift to the right of wind observed by Fridtjof Nansen.

4) As the transport is 90° to the right of the wind vector, near shore upwelling is attributed in the coastal waters when the wind blows parallel to the coast with land mass to the left of the wind vector, like Waltair coast in summer (Lafond, 1954). It may be noted that wind set parallel to the coast need not produce upwelling unless other conditions are fulfilled. All those conditions or assumptions culminated to

$\mu_z = u_o \exp(-\pi Z/D) \cos(\pi/4 - \pi^Z/D)$

$V_2 = u_o \exp(-\pi Z/D) \sin(\pi/4 - \pi\, Z/D)$

$u_o = {}^1/2n\, (d^2\, V/\partial\, Z^2)z{=}o$

Where $n = \rho f/2\mu$

$(\tan \Phi)_{z=o} = \pi/4 = 45°$

And the current crosses x axis in its veering at z = D/4. Therefore, until z is one fourth D, the veering currents remain in the first quadrant and then enter into the 4th quadrant.

Orientation of frictional force in the Ekman model:

It would be misleading if we adopt the general convention of direction of frictional force acting on a particle or parcel, in motion at a point, as it would not lead to balance of frictional force with the coriolis force. Physically the coriolis force is balanced by vertical divergence of momentum flux which is given by μ/ρ times the second vertical derivative of velocity. To prove that the second vertical derivative of velocity is at right angles to velocity, we take velocity components expressed in eqs. 12 & 13 in Cartesian system of coordinates and perform the second derivatives of them which are obtained as $\partial^2 u/\partial z^2 = -2(\pi/D)^2 \exp(-\pi Z/D) \sin(\pi/4 - \pi z/D)$ 21

and $\partial^2 v/\partial z^2 = -2(\pi/D)^2 \exp(-\pi Z/D) \cos(\pi/4 - \pi Z/D)$ 22

and obtain the dot product (Pipes, 1958) with the velocity vector.

By vector analysis

$V = iu + j\,v$ 23

$\partial^2 v/\partial z^2 = i\,\partial^2 u/\partial z^2 + j\,\partial^2 v/\partial z^2$ 24

where i and j are unit vectors along x & y respectively. Dot product of the above vectors is

$V.\ \partial^2 V/\partial z^2 = u\partial^2 u/\partial z^2 + V\partial^2 v/\partial z^2$ 25

From eqs. 12,13,21 & 22 the dot product of the velocity vector and vector second vertical derivative of velocity is zero proving that the vectors V and $\partial^2 v/\partial z^2$ are at right angles to one another (see Fig.3).

Fig.3 schematically represents the balance of coriolis force OD and the frictional force OB at a point O at a depth Z along the vertical axis. Both the forces are at right angles to the velocity vectior V. In the triangle OAB, OB is the frictional force vector which is resolved into its components OA and AB. Similarly OD is the coriolis force with resolved components OC and CD. It may be noticed that fv balances F_x and fu balance Fy where $F_x = (\mu/\rho)\,\frac{\delta^2 u}{\delta z^2}$ and $Fy = (\mu/p)d^2 v/dz^2$.

Coastal water upwelling :

Horizontal divergence produced and maintained by surface winds near the coast is compensated by horizontal convergence in sub-surface layers due to poleward transport of water, thus leading to the upward transport of water from lower layers to the surface layer. This ascent of water from lower layers takes place in to the surface layer, even up to the very surface (Yoshida and Mao, 1957). Prof. Hidaka (1954) also assumed that the vertical currents reduce to zero only at the surface of the sea. However, upward velocities should reduce to zero in the core of thermocline which becomes more intensified hence more stable during the season of upwelling (Lathipha and Murty, 1978). As a result of this, chemical concentrations and other properties like low temperature are exchanged from the depth where upwelling ceases into the top including surface layer by eddy diffusion only.

If the wind blows parallel to the coast leaving the coast to its left, mean transport of water from the Ekman spiral makes

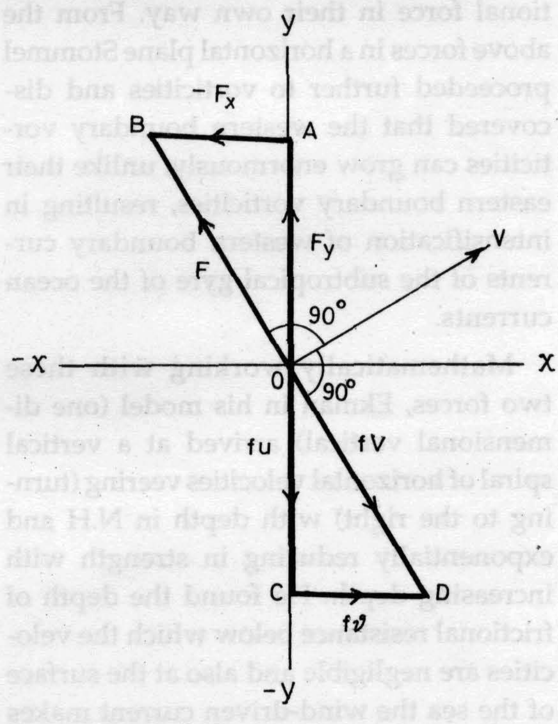

Fig.3 Balance of frictional force and coriolis force in the Ekman spiral

V -velocity vector, F - frictional force, Fy & Fx tis components, fV-Coriolis force, fu & fv its components.

90° to the coastline (Sverdrup *et. al.*, 1942). This necessitates convergence at lower layer to compensate the out flow from the spiral. Thus the spiral gives room for upwelling process to take place near the coast. However, due to the single dimensional nature of the Ekman spiral and as upwelling covers over a wider space, Ekman spiral has got limited effect in producing coastal upwelling. Nevertheless, Ekman theory remains as a classical methematical theory with limited applications, unlike Stommel's theory.

Epilogue

Henry Stommel and Walfrid Ekman both considered coriolis force and frictional force in their own way. From the above forces in a horizontal plane Stommel proceeded further to vorticities and discovered that the western boundary vorticities can grow enormously, unlike their eastern boundary vorticities, resulting in intensification of western boundary currents of the subtropical gyre of the ocean currents.

Mathematically working with these two forces, Ekman in his model (one dimensional vertical) arrived at a vertical spiral of horizontal velocities veering (turning to the right) with depth in N.H and exponentially reducing in strength with increasing depth. He found the depth of frictional resistance below which the velocities are negligible and also at the surface of the sea the wind-driven current makes an angle of 45° to the right of wind stress.

Stommel followed the general convention that frictional force works out in a direction opposite to the direction of velocity. In the Ekman model, the second vertical derivative of velocity vector multiplied by u/p gives the frictional force opposing coriolis force to balance.

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